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Ultradistributions and Hyperfunctions (超函数と解析汎函 数の理論と応用)

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Ultradistributions and hyperfunctions

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In the last conference of March, 1971, the speaker announced the following theorem and applied it to the theory of ordinary differential equations with real analytic coefficients.

Theorem. Let $f = [F]$ be a hyperfunction on an interval (a, b) with a defining function F . Then f is an ultradistribution of Gevrey class of order s of Roumieu type (of Beurling type) if and only if for every compact interval $K \subset (a, b)$ and every $L > 0$ there is a constant C (there are constants L and C) such that

$$\sup_{x \in K} |\varphi(x + iy)| \leq C \exp \left\{ \left(\frac{L}{|y|} \right)^{\frac{1}{s-1}} \right\}$$

In this lecture we develop the theory of ultradistributions and give a proof of the theorem in a generalized form.

1. Ultradifferentiable functions. Let M_p , $p = 0, 1, 2, \dots$, be a sequence of positive numbers. An infinitely differentiable function f on an open set Ω in \mathbb{R}^n will be called an ultradifferentiable function of class M_p of Roumieu type (of Beurling type) if for every compact set K in Ω there are constants h and C (and for every $h > 0$ there is a constant C) such that

$$(1) \quad \|D^\alpha f\|_{C(K)} \leq C h^p M_p, \quad |\alpha| = p = 0, 1, 2, \dots$$

We will impose the following conditions on M_p :

(M.1) (Logarithmic convexity)

$$(2) \quad M_p^2 \leq M_{p-1} M_{p+1}, \quad p = 1, 2, \dots$$

(M.2) (Stability under convolution) There are constants A and H such that

$$(3) \quad M_p \leq A H^p \min_{0 \leq q \leq p} M_{p-q} M_q, \quad p = 0, 1, 2, \dots$$

(M.3) (Strong non-quasi-analyticity) There is a constant A such that

$$(4) \quad \sum_{j=p}^{\infty} \frac{M_j}{M_{j+1}} \leq A \frac{p M_p}{M_{p+1}}, \quad p = 1, 2, 3, \dots$$

In some problems (M.2) and (M.3) may be replaced by the following weaker conditions:

(M.2)' (Stability under differentiation)

$$(5) \quad M_{p+1} \leq A H^p M_p, \quad p = 0, 1, 2, \dots$$

(M.3)' (Non-quasi-analyticity)

$$(6) \quad \sum_{j=0}^{\infty} \frac{M_j}{M_{j+1}} < \infty$$

It is easy to check that the Gevrey sequences

$$(7) \quad M_p = (p!)^s, \quad p^{sp} \quad \text{and} \quad \Gamma(1+sp),$$

where $s > 1$, satisfy these conditions. These sequences determine the same class of ultradifferentiable functions called the Gevrey class of order s .

It is convenient to relate the above conditions with the behavior of the associated function

$$(8) \quad M(\rho) = \log \sup_p \frac{\rho^{p M_0}}{M_p}.$$

(M.1) is equivalent to

$$(9) \quad \frac{M_p}{M_0} = \sup_{\rho > 0} \frac{\rho^p}{\exp M(\rho)}.$$

Under this condition

$$(10) \quad M(\rho) = \int_0^\rho \frac{m(\lambda)}{\lambda} d\lambda,$$

where $m(\lambda)$ is the number of ratios $m_j = M_j/M_{j-1}$ which does not exceed λ .

(M.2) is equivalent to

$$(11) \quad 2M(\rho) \leq M(H\rho) + \log(AM_0).$$

(M.3) implies

$$(12) \quad \rho \int_\rho^\infty \frac{m(\lambda)}{\lambda^2} d\lambda \leq AM(\rho).$$

On the other hand, (M.2)' is equivalent to

$$(13) \quad m(\lambda) \geq \frac{\log(\lambda/A')}{\log H}.$$

(M.3)' is equivalent to

$$(14) \quad \int_0^\infty \frac{M(\rho)}{\rho^2} d\rho = \int_0^\infty \frac{m(\lambda)}{\lambda^2} d\lambda < \infty.$$

Definition 1. Let K be a compact set in \mathbb{R}^n and $h > 0$. We denote by $\mathcal{E}_{\{M_p\},h}^{(regular)}(K)$ the Banach space of all functions $f \in C^\infty(K)$ in the sense of Whitney such that

$$(15) \quad \|f\|_{\mathcal{E}_{\{M_p\},h}(K)} = \sup_{\alpha, x} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty,$$

and by $\mathcal{D}_K^{\{M_p\},h}$ the Banach space of all functions $f \in C^\infty(\mathbb{R}^n)$ with support in K which satisfies (15).

$\mathcal{D}_K^{\{M_p\},h}$ may be looked upon as a closed subspace of $\mathcal{E}_{\{M_p\},h}^{(regular)}(K)$.

Proposition 2. If $h < k$, the injections

$$(16) \quad \mathcal{E}_{\{M_p\},h}^{(regular)}(K) \subset \mathcal{E}_{\{M_p\},k}^{(regular)}(K)$$

$$(17) \quad \mathcal{D}_K^{\{M_p\},h} \subset \mathcal{D}_K^{\{M_p\},k}$$

are compact. If M_p satisfies (M.2)' in addition and if k/h is sufficiently large, then the injections are nuclear.

Definition 3. Let K be a regular compact set and Ω an open set in \mathbb{R}^n . We define the spaces of ultradifferentiable functions of Roumieu type $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{E}^{\{M_p\}}(\Omega)$ and those of Beurling type $\mathcal{E}^{(M_p)}(K)$ and $\mathcal{E}^{(M_p)}(\Omega)$ by

$$(18) \quad \mathcal{E}^{\{M_p\}}(K) = \varinjlim_{h \rightarrow \infty} \mathcal{E}^{\{M_p\},h}(K),$$

$$(19) \quad \mathcal{E}^{\{M_p\}}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}(K),$$

$$(20) \quad \mathcal{E}^{(M_p)}(K) = \varprojlim_{h \rightarrow 0} \mathcal{E}^{\{M_p\},h}(K),$$

$$(21) \quad \mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{E}^{(M_p)}(K).$$

It follows from Proposition 2 that $\mathcal{E}^{\{M_p\}}(K)$ is a (DFS)-space and $\mathcal{E}^{(M_p)}(K)$ and $\mathcal{E}^{(M_p)}(\Omega)$ are (FS)-spaces. If M_p satisfies (M.2)', these spaces are all nuclear.

Similarly the spaces of ultra-differentiable functions with compact support are defined in the following way:

$$(22) \quad \mathcal{D}_K^{\{M_p\}} = \varinjlim_{h \rightarrow \infty} \mathcal{D}_K^{\{M_p\},h},$$

$$(23) \quad \mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K \Subset \Omega} \mathcal{D}_K^{\{M_p\}},$$

$$(24) \quad \mathcal{D}_K^{(M_p)} = \varprojlim_{h \rightarrow 0} \mathcal{D}_K^{\{M_p\},h},$$

$$(25) \quad \mathcal{D}^{(M_p)}(\Omega) = \varprojlim_{K \Subset \Omega} \mathcal{D}_K^{(M_p)}.$$

$\mathcal{D}_K^{\{M_p\}}$ and $\mathcal{D}^{\{M_p\}}(\Omega)$ are (DFS)-spaces, $\mathcal{D}_K^{(M_p)}$ is an (FS)-space and $\mathcal{D}^{(M_p)}(\Omega)$ is an (LF)-space as the strict inductive limit of a sequence of (FS)-spaces. Hence all spaces are Hausdorff, complete, reflexive and bornologic. If M_p satisfies (M.2)', then all spaces are nuclear.

A subset B of $\mathcal{D}_K^{\{M_p\}}$ or $\mathcal{D}^{\{M_p\}}(\Omega)$ is bounded if and only if it is contained in a $\mathcal{D}_K^{\{M_p\},h}$ and bounded there, while a subset B of $\mathcal{D}_K^{(M_p)}$ or $\mathcal{D}^{(M_p)}(\Omega)$ is bounded if and only if it is contained in a $\mathcal{D}_K^{(M_p)}$ for a K and bounded in all $\mathcal{D}_K^{\{M_p\},h}$.

It is well known that 1) $\mathcal{D}_K^{\{M_p\}} = \mathcal{D}_K^{\{M'_p\}}$, where M'_p is the greatest logarithmically convex sequence such that $M'_p \leq M_p$ and that in case M_p is logarithmically convex, $\mathcal{D}_K^{\{M_p\}} \neq 0$ if and only if M_p satisfies (M.3)'. Conversely suppose that M_p satisfies (M.1) and (M.3)'. Then for any ball K of radius $\varepsilon > 0$ there is a function $\rho_\varepsilon \in \mathcal{D}_K^{\{M_p\}}$ such that $\rho_\varepsilon(x) \geq 0$ and $\int \rho(x) dx = 1$. Hence it follows that $\mathcal{D}^{\{M_p\}}(\Omega)$ is dense in $\mathcal{D}(\Omega)$ and that there exists a partition of unity by functions in $\mathcal{D}^{\{M_p\}}(\Omega)$ subordinate to any open covering of Ω .

If M_p satisfies (M.1) and (M.3)', there is M'_p which satisfies (M.1), (M.3)' and

$$(26) \quad \lim_{p \rightarrow \infty} \frac{h^p M_p}{M'_p} = 0 \quad \text{for any } h > 0.$$

Thus the same results as above hold for $\mathcal{D}^{(M_p)}(\Omega)$.

1) See Mandelbrojt [8], [9], Roumieu [10], [11] and Lions-Magenes [7] for the results up to the end of this section.

If M_p satisfies (M.1), the spaces $\mathcal{E}_p^{\{M\}}(K)$, $\mathcal{E}_p^{\{M\}}(\Omega)$, $\mathcal{E}_p^{(M)}(K)$ and $\mathcal{E}_p^{(M)}(\Omega)$ are stable under multiplication and the spaces $\mathcal{D}_K^{\{M\}}$, $\mathcal{D}_p^{\{M\}}(\Omega)$, $\mathcal{D}_K^{(M)}$ and $\mathcal{D}_p^{(M)}(\Omega)$ are stable under multiplication by functions in $\mathcal{E}_p^{\{M\}}(K)$, $\mathcal{E}_p^{\{M\}}(\Omega)$, $\mathcal{E}_p^{(M)}(K)$ and $\mathcal{E}_p^{(M)}(\Omega)$ respectively and the multiplications are hypo-continuous.

If (M.2)' holds, the above spaces are stable under differentiation and D^α is continuous for any α .

The spaces of Roumieu type have been discussed by Roumieu [10] and [11]. However, it is not clear whether or not the topologies he employed coincide with the above natural topologies which have been introduced by Lions-Magenes [7].

The spaces of Beurling type have been discussed in Björck [1] from a little different point of view and in Lions-Magenes [7].

2. The Paley-Wiener theorem for ultra-differentiable functions.

Theorem 4. Suppose that M_p satisfies (M.1) and (M.2)' and that K is a compact convex set in \mathbb{R}^n . Then a function $\varphi(x)$ belongs to $\mathcal{D}_K^{\{M\}}$ ($\mathcal{D}_K^{(M)}$) if and only if there are h and C (for any $h > 0$ there is C) such that the Fourier-Laplace transform

$$(27) \quad \tilde{\varphi}(\zeta) = \mathcal{F}\varphi(\zeta) = \int_{\mathbb{R}^n} e^{ix\zeta} \varphi(x) dx$$

of φ satisfies

$$(28) \quad |\tilde{\varphi}(\zeta)| \leq C \exp(-M(|\zeta|/h) + H_K(\zeta)),$$

where

$$(29) \quad H_K(\zeta) = \sup_{x \in K} \operatorname{Im} \langle x, \zeta \rangle.$$

A subset B of $\mathcal{D}_K^{\{M_p\}}$ $(\mathcal{D}_K^{(M_p)})$ is bounded if and only if
we can choose constants h and C (for any $h > 0$ a constant $C)$
uniformly for $\varphi \in B$.

A sequence of functions $\varphi_j \in \mathcal{D}_K^{\{M_p\}}$ $(\mathcal{D}_K^{(M_p)})$ converges if
and only if for some $h > 0$ (for any $h > 0)$ $\exp M(|\zeta|/h) \tilde{\varphi}_j(\zeta)$
converges uniformly on \mathbb{R}^n or equivalently on a strip $|\operatorname{Im} \zeta| < a$,
where $0 < a < \infty$.

Since $\mathcal{D}_K^{(M_p)}$ is a Fréchet space, this shows that the families
of semi-norms

$$(30) \quad \sup_{\zeta \in \mathbb{C}^n} \exp(M(k|\zeta|) - H_K(\zeta)) \tilde{\varphi}(\zeta), \quad k = 1, 2, \dots$$

and

$$(31) \quad \sup_{\zeta \in \mathbb{R}^n} \exp M(k|\zeta|) \tilde{\varphi}(\zeta), \quad k = 1, 2, \dots$$

determine the topology of $\mathcal{D}_K^{(M_p)}$.

In order to find a family of semi-norms similar to (30) or
(31) which determines the topology of $\mathcal{D}_K^{\{M_p\}}$, we imbed the
Fourier-Laplace transform of $\mathcal{D}_K^{\{M_p\}}$ in a (DFS^*) -space.

Let $1 < r < \infty$ be fixed and consider the sequence of Banach
spaces

$$(32) \quad Y_h = \{ \psi \in L^r_{loc}(\mathbb{C}^n); \exp(M(|\zeta|/h) - H_K(\zeta)) \psi(\zeta) \in L^r(\mathbb{C}^n) \},$$

$$h = 1, 2, \dots$$

with the identity mappings $Y_h \rightarrow Y_{h+1}$. Since Y_h are reflexive Banach spaces, this forms a weakly compact sequence and its limit $Y = \varinjlim Y_h$ is a (DFS*)-space.

A modified form of Morera's theorem shows that

$$(33) \quad X_h = \{ \psi \in Y_h; \psi \text{ is entire on } \mathbb{C}^n \}$$

is a closed subspace of Y_h . We can prove that

$$(34) \quad \mathcal{H}_K^{\{M\}_P} = \varinjlim_{h \rightarrow \infty} X_h$$

including the topology. Morera's theorem proves also that the set

$$\mathcal{H}_K^{\{M\}_P} \text{ is closed in } Y \text{ and that } X_h = Y_h \cap \mathcal{H}_K^{\{M\}_P}.$$

Since $\mathcal{D}_K^{\{M\}_P}$ is a Montel space, it is proved that the original topology of $\mathcal{H}_K^{\{M\}_P}$ induced by that of $\mathcal{D}_K^{\{M\}_P}$ coincides with the relative topology induced by that of Y (cf. [5] Theorem 7).

Theorem 5. Under the same assumptions as in Theorem 4 the topology of $\mathcal{H}_K^{\{M\}_P}$ is determined by the family of semi-norms

$$(35) \quad \sup_{\zeta \in \mathbb{C}^n} | \exp(M(\varepsilon(|\zeta|)) - H_K(\zeta)) \tilde{\varphi}(\zeta) |$$

when $\varepsilon(\rho)$ runs through the increasing functions on $[0, \infty)$ satisfying

$$(36) \quad \lim_{\rho \rightarrow \infty} \frac{\varepsilon(\rho)}{\rho} = 0.$$

From the Paley-Wiener theorem (Theorem 4) we get easily the

following

Theorem 6. Suppose that M_p satisfies (M.1), (M.2) and (M.3)'.
Let

$$(37) \quad J(\zeta) = \sum_{|\alpha|=0}^{\infty} a_{\alpha} \zeta^{\alpha}$$

be an entire function with the growth order that for any $L > 0$

there is C (there are L and C) such that

$$(38) \quad |J(\zeta)| \leq C \exp M(L|\zeta|), \quad \zeta \in \mathbb{C}^n.$$

Then, for any compact convex set K in \mathbb{R}^n the differential operator of infinite order

$$(39) \quad J(D) = \sum_{|\alpha|=0}^{\infty} a_{\alpha} D^{\alpha}$$

maps $\mathcal{D}_K^{\{M_p\}}(\mathcal{D}_K^{(M_p)})$ continuously into itself. Moreover, the right hand side of

$$(40) \quad J(D) \varphi(x) = \sum_{|\alpha|=0}^{\infty} a_{\alpha} D^{\alpha} \varphi(x)$$

converges absolutely in the topology of $\mathcal{D}_K^{\{M_p\}}(\mathcal{D}_K^{(M_p)})$ and (40) holds for any $\varphi \in \mathcal{D}_K^{\{M_p\}}(\mathcal{D}_K^{(M_p)})$. More precisely if φ is contained in a bounded set of $\mathcal{D}_K^{\{M_p\}}(\mathcal{D}_K^{(M_p)})$, the partial sums of (40) are contained in an absolutely convex bounded set B and the series converges absolutely in the normed space generated by B .

An entire function $J(\zeta)$ satisfying (38) will be called a multiplier for the class $\{M_p\}(\{M_p\})$. It is easy to see that (37) is a multiplier for $\{M_p\}(\{M_p\})$ if and only if for any $L > 0$ there is C (there are L and C) such that

$$(41) \quad |a_{\alpha}| \leq C L^{|\alpha|} / M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \dots$$

Proposition 7. Suppose that M_p satisfies (M.1), (M.2) and

(M.3). Then an entire function $J(\zeta)$ of one variable is a multiplier for $\{M_p\}$ ((M_p)) if and only if it has Hadamard's factorization ([2], p.22)

$$(42) \quad J(\zeta) = a \zeta^{n_0} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{c_j}\right)$$

and for any $L > 0$ there is C (there are L and C) such that

$$(43) \quad N(\rho) = \int_0^\rho \frac{n(\lambda) - n_0}{\lambda} d\lambda$$

$$\leq M(L\rho) + \log C, \quad 0 < \rho < \infty,$$

where $n(\lambda)$ is the number of c_j with $|c_j| \leq \lambda$.

Finally we obtain a characterization of the Fourier-Laplace transforms of ultra-differentiable functions with compact support in a way similar to Ehrenpreis [3]. Since (40) converges absolutely in the original topology, ours may be said a better characterization.

Theorem 8. Suppose that M_p satisfies (M.1), (M.2) and (M.3) and that K is a compact convex set in \mathbb{R}^n . Then a function $\varphi(x)$ belongs to $\mathcal{D}_K^{\{M_p\}}$ ($\mathcal{D}_K^{(M_p)}$) if and only if its Fourier-Laplace transform $\tilde{\varphi}(\zeta)$ satisfies

$$(44) \quad \sup_{\zeta} |\exp(-H_K(\zeta)) J(\zeta) \tilde{\varphi}(\zeta)| < \infty$$

for any entire function $J(\zeta)$ of the form

$$(45) \quad J(\zeta) = J_0(s_1 \zeta_1) \cdots J_0(s_n \zeta_n),$$

$$(46) \quad J_0(\zeta) = \prod_{j=1}^{\infty} \left(1 + \frac{l_j \zeta}{m_j}\right),$$

where s_j is +1 or -1 and l_j is a sequence of positive numbers converging to zero (l_j is a positive constant).

Moreover, the family of semi-norms (44) determines the

topology of $\mathcal{D}_K^{\{M_p\}} (\mathcal{D}_K^{(M_p)})$.

3. Ultra-distributions.

Definition 9. Suppose that M_p satisfies (M.1) and (M.3)' and that Ω is an open set in \mathbb{R}^n . We denote by $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$) the strong dual space of $\mathcal{D}^{\{M_p\}}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) and call its elements ultra-distributions on Ω of class M_p of Roumieu type (Beurling type) or of class $\{M_p\}$ ((M_p)) for short.

Since $\mathcal{D}^{\{M_p\}}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) is a dense subspace of $\mathcal{D}(\Omega)$ and the injection is continuous, $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$) contains the distributions $\mathcal{D}'(\Omega)$ as a dense subspace.

On the other hand, since the real analytic functions on Ω are continuously and densely contained in $\mathcal{E}^{\{M_p\}}(\Omega)$ ($\mathcal{E}^{(M_p)}(\Omega)$), it follows that every ultra-distribution is a hyperfunction.

If $a \in \mathcal{E}^{\{M_p\}}(\Omega)$ ($\mathcal{E}^{(M_p)}(\Omega)$) and $f \in \mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$), the product af is defined by

$$(47) \quad \langle af, \varphi \rangle = \langle f, a\varphi \rangle, \quad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega) \text{ } (\mathcal{D}^{(M_p)}(\Omega)).$$

If M_p satisfies (M.2)', the derivative $D^\alpha f$ is defined by

$$(48) \quad \langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \quad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega) \text{ } (\mathcal{D}^{(M_p)}(\Omega)).$$

Similarly if M_p satisfies (M.2) and $J(\zeta)$ is a multiplier for $\{M_p\}$ ((M_p)), $J(D)f$ is defined by

$$(49) \quad \langle J(D)f, \varphi \rangle = \langle f, J(-D)\varphi \rangle, \quad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega) \text{ } (\mathcal{D}^{(M_p)}(\Omega)).$$

As in the case of distributions, the existence of partition of unity implies that $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)'}(\Omega)$), $\Omega \subset \mathbb{R}^n$, with

the natural restriction mappings forms a soft sheaf on \mathbb{R}^n . In particular, the notion of support is defined.

If S is a closed set in Ω , the subspace of all ultra-distributions f in $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) with $\text{supp } f \subset S$ is closed.

Multiplications $a \cdot$, differentiations D^α and $J(D)$ are sheaf homomorphisms. Namely they do not enlarge the support.

The dual space $\mathcal{E}^{\{M_p\}'}(\Omega)$ ($\mathcal{E}^{(M_p)}(\Omega)$) of $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) is identified with the subspace composed of all $f \in \mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) with compact support.

Theorem 10. Suppose that M_p satisfies (M.1), (M.2)' and (M.3)'. Then, a hyperfunction f on an open set Ω in \mathbb{R}^n belongs to $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) if and only if its restriction $f|_G$ to any relatively compact open set G in Ω can be written

$$(50) \quad f|_G = \sum_{|\alpha|=0}^{\infty} D^\alpha f_\alpha,$$

where $f \in C(\bar{G})'$ or $L^r(G)$, $1 \leq r \leq \infty$, and for any $L > 0$ there is C (there are L and C) such that

$$(51) \quad \|f_\alpha\| \leq C \frac{L^{|\alpha|}}{M_{|\alpha|}^{\{M_p\}'}}.$$

(50) converges strongly in $\mathcal{D}^{\{M_p\}'}(G)$ ($\mathcal{D}^{(M_p)}(G)$).

A subset B of $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{D}^{(M_p)}(\Omega)$) is bounded if and only if constant(s) C (and L) in (51) can be chosen uniformly in $f \in B$.

Roumieu [10], Chap.I, théorème 1 gives a stronger statement.

By the Phragmen-Lindelöf theorem we can show that the semi-norm (44) in Theorem 8 is equivalent to

$$\sup_{\xi \in \mathbb{R}^n} |J(\xi) \tilde{\varphi}(\xi)|.$$

Hence we obtain another structure theorem:

Theorem 11. Suppose that M_p satisfies (M.1), (M.2) and (M.3). Then, a hyperfunction f on an open set Ω in \mathbb{R}^n belongs to $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{E}^{(M_p)'}(\Omega)$) if and only if for any relatively compact convex open set G there is a multiplier $J(\zeta)$ for the class $\{M_p\}$ ((M_p)) and a finite measure f on G such that

$$(52) \quad f|_G = J(D)f.$$

A subset B of $\mathcal{D}^{\{M_p\}'}(\Omega)$ ($\mathcal{E}^{(M_p)'}(\Omega)$) is bounded if and only if there is $J(D)$ independent of $f \in B$ and $\|f\|$ are bounded.

4. Characterization of ultra-distributions. In this section we consider only the case where $n = 1$ for the sake of simplicity.

When M_p is a sequence satisfying (M.1), we write

$$(53) \quad M^*(\rho) = \log \sup_p \frac{\rho^p p! M_0}{M_p},$$

$$(54) \quad M_p^* = M_0 \sup_{\rho > 0} \frac{\rho^p}{\exp M^*(\rho)}.$$

If m_p/p is increasing, we have $M_p^* = M_p/p!$.

Theorem 12. Suppose that M_p satisfies (M.1), (M.2) and (M.3). Then, a hyperfunction $f = [F]$ on an interval (a, b) belongs to $\mathcal{D}^{\{M_p\}'}(a, b)$ ($\mathcal{E}^{(M_p)'}(a, b)$) if and only if for any compact

interval K in (a, b) and for any $L > 0$ there is C (there are L and C) such that the defining function F satisfies

$$(55) \quad \sup_{x \in K} |F(x+iy)| \leq C \exp M^* \left(\frac{L}{|y|} \right)$$

for sufficiently small $|y|$.

A subset B of $\mathcal{O}_{\{M_P\}}^{(M_P)'}(a, b)$ ($\mathcal{O}_{\{M_P\}}^{(M_P)'}(a, b)$) is bounded if and only if the constant(s) C (and L) can be chosen uniformly in $f \in B$.

Sketch of Proof. Suppose that F satisfies (55) for $K = [c, d]$. We will find multipliers $J_+(\zeta)$ and $J_-(\zeta)$ and holomorphic functions G_+ and G_- which are bounded near (c, d) such that

$$(56) \quad F(x+iy) = \begin{cases} J_+(D)G_+(x+iy), & y > 0 \\ J_-(D)G_-(x+iy), & y < 0. \end{cases}$$

Then $f = J_+(D)G_+(x+i0) - J_-(D)G_-(x-i0)$ belongs to $\mathcal{O}_{\{M_P\}}^{(M_P)'}(c, d)$ ($\mathcal{O}_{\{M_P\}}^{(M_P)'}(c, d)$).

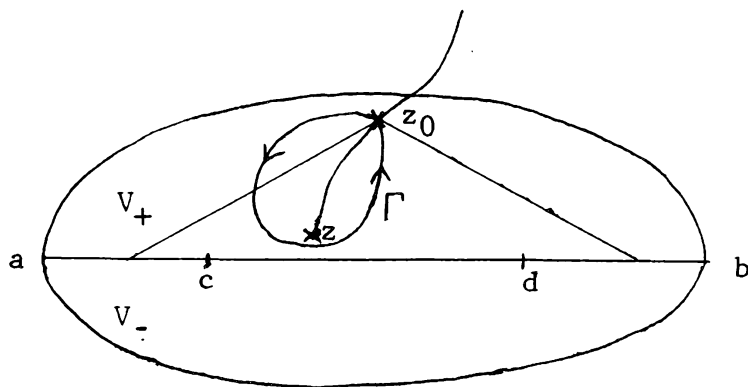
Let $y > 0$ and

$$(57) \quad J_+(\zeta) = (1+\zeta)^2 \prod_{j=1}^{\infty} \left(1 + \frac{l_j \zeta}{m_j} \right),$$

where l_j is a positive sequence converging to zero (a positive constant) Since $J_+(\zeta)^{-1}$ is infra-exponential except on the negative real axis,

$$(58) \quad G_+(z) = \frac{1}{2\pi} \int_0^{\infty} e^{i\alpha x} J_+(\zeta)^{-1} e^{iz\zeta} d\zeta$$

defines a holomorphic function on the Riemann surface $-\pi < \arg z < 2\pi$.



Choose a point z_0 in the upper domain V_+ of F and define for z in the cone $-\pi + \varepsilon < \arg(z - z_0) < -\varepsilon \subset V_+$

$$(59) \quad G_+ F(z) = \int_{\Gamma} G_+(z-w) F(w) dw ,$$

where Γ is a simple closed curve starting z_0 and encircling the slit $[z, z_0]$ counterclockwise. Then we have

$$(60) \quad J_+(D) G_+ F(z) = F(z) .$$

By deforming the contour Γ we have

$$(61) \quad G_+ F(z) = i \int_0^t g_+(-iy) F(z+iy) dy + \dots ,$$

where

$$(62) \quad g_+(-iy) = \frac{1}{2\pi i} \int_{\xi-i\infty}^{\xi+i\infty} J_+(\xi+i\eta)^{-1} e^{y(\xi+i\eta)} d\eta .$$

Taking it into account that

$$(63) \quad |g(-iy)| \leq \frac{1}{2\pi} \left| \int_{-i\infty}^{i\infty} \frac{d\zeta}{(1+\zeta)^2} \right| \inf_{\xi>0} \frac{e^{y\xi}}{\prod \left| 1 + \frac{l_j \xi}{m_j} \right|} \\ \leq C \inf_{\xi>0} \frac{e^{y\xi}}{\exp \tilde{M}(\xi)} ,$$

where

$$(64) \quad \tilde{M}(\xi) = \log \sup_p \frac{l_1 \cdots l_p \xi^{P_{M_0}}}{M_p} ,$$

we can choose a sequence l_j so that the first term of (61) is

bounded. The remainder is also bounded. Hence we have (56).

The proof shows that if the estimate (55) is uniform in $f \in B$, then $J_{\pm}(D)^{-1}$ constructed above map $F(x \pm i0)$ into a bounded set in $L^{\infty}(c, d)$, and hence B is bounded.

Conversely suppose that $f \in \mathcal{O}_{\{M_p\}}^{(M_p)'}(a, b) \cap (\mathcal{O}_{\{M_p\}}^{(M_p)'}(a, b))$.

It follows from Theorem 10 that

$$f|_{(c, d)} = \sum_{p=0}^{\infty} D^p f_p, \quad f_p \in C([c, d])' \quad \text{and} \\ \|f_p\|_{C([c, d])'} \leq C \frac{L^p}{M_p}.$$

Let F_p be the standard defining function of f_p . Then we have the estimate

$$\sup_{x \in [c, d]} |D^p F_p(x + iy)| \leq \frac{C}{2\pi} \frac{L^p p!}{|y|^{p+1} M_p} \\ \leq \frac{1}{2^p} \frac{CA}{2\pi M_0} \sup_p \frac{(2HL)^{p+1} (p+1)! M_0}{|y|^{p+1} M_{p+1}}.$$

Therefore

$$F(x + iy) = \sum_{p=0}^{\infty} D^p F_p(x + iy)$$

is a defining function of f and it satisfies

$$\sup_{x \in [c, d]} |F(x + iy)| \leq \frac{CA}{\pi M_0} \exp M^*\left(\frac{2HL}{|y|}\right).$$

It is clear that if a defining function satisfies (55), any other defining function satisfies it also.

For the Gevrey sequence of order s , $M^*(\rho)$ is equivalent to $\rho^{\frac{1}{s-1}}$. Therefore the theorem in the introduction is a special case of Theorem 12.

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